Journal of Differential Equations **173**, 145–159 (2001)
doi:10.1006/jdeq.2000.3868, available online at http://www.idealibrary.com on IDEAL®

The Nodal Surface of the Second Eigenfunction of the Laplacian in $\mathbf{R}^D$ Can Be Closed

Søren Fournais[1]

Department of Mathematics, Ny Munkegade, 8000 Aarhus C, Denmark
E-mail: fournais@imf.au.dk

Received September 28, 1999; revised March 22, 2000

We construct a set in $\mathbf{R}^D$ with the property that the nodal surface of the second eigenfunction of the Dirichlet Laplacian is closed, i.e. does not touch the boundary of the domain. The construction is explicit in all dimensions $D \geqslant 2$ and we obtain

1. INTRODUCTION

It is a famous conjecture by Payne [8] that the nodal surface of the 2nd eigenfunction of the Dirichlet Laplacian on a bounded domain Ω in $\mathbf{R}^2$ touches the boundary $\partial\Omega$. The conjecture was later extended by S. T. Yau to higher dimension [12]. For convex domains in $\mathbf{R}^2$ the original conjecture was proved by Melas [7]. Later, in [5] the conjecture was proved for another class of domains that are not necessarily simply connected. Furthermore D. Jerison has proved the conjecture in any dimension for long thin convex domains and, in that case, obtained information on the position of the nodal set [6]. However, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili [4] constructed a non-simply connected counterexample to the general conjecture in $\mathbf{R}^2$. The 2-dimensional example in [4] relies heavily on choosing a very symmetric domain and using symmetry arguments. In higher dimensions it is not possible to choose similar, very symmetric domains. The obstruction being that there are only a finite number of regular polyhedra in any dimension greater than or equal to 3 (in 3 dimensions these are the platonic solids). In this paper we will look at a natural higher dimensional generalisation of the domain in [4]. Because of the lack of symmetries the argument from [4] cannot be applied. We shall use an alternative, and in a way more direct, argument to reach the desired conclusion.

[1] Partially supported by the European Union, Grant FMRX-960001.

0022-0396/01 $35.00

1.1. *Generalities*

For a bounded connected domain Ω (with sufficiently regular boundary) we will look at the Laplace operator with Dirichlet conditions at the boundary. This defines a positive, self adjoint operator $-\Delta_\Omega$ with domain $W_0^{1,2}(\Omega)$ (see [3] for notation) and purely discrete spectrum. We denote the eigenvalues (eigenfunctions) by $\{\lambda_j(\Omega)\}_{j=1}^\infty$ ($\{u_j(\Omega)\}_{j=1}^\infty$), so

$$\begin{aligned} -\Delta_\Omega u_j(\Omega) &= \lambda_j(\Omega)\, u_j(\Omega) && \text{in } \Omega \\ u_j(\Omega) &\equiv 0 && \text{on } \partial\Omega, \end{aligned}$$

and the eigenvalues are ordered according to size: $0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leqslant \lambda_3(\Omega) \leqslant \cdots$ (it is a general result that $\lambda_1(\Omega)$ is simple and strictly positive). We may take the eigenfunctions to be real and orthonormal: $\langle u_j(\Omega), u_k(\Omega)\rangle = \delta_{j,k}$, where δ is the Kronecker delta. Since the eigenfunctions are real, and the first can be chosen positive, the second eigenfunction $u_2(\Omega)$ has to take both positive and negative values. According to Courant's Nodal Domains Theorem Ω splits into exactly two connected open sets Ω_+, Ω_- such that $u_2 > 0$ on Ω_+, $u_2 < 0$ on Ω_- and $\bar{\Omega} = \bar{\Omega}_+ \cup \bar{\Omega}_-$.

It is now natural to study the geometry of the nodal set $\mathcal{N}(u_2)$, where

$$\mathcal{N}(u_2) = \overline{\{x \in \Omega \mid u_2(\Omega)(x) = 0\}}.$$

Generically (see [11]), this is a manifold of codimension 1, and one may ask whether it always touches the boundary of the domain, i.e. whether

$$\mathcal{N}(u_2) \cap \partial\Omega \neq \emptyset$$

always. This is the above mentioned conjecture by Payne [8].

1.2. *The domain*

We choose $0 < R_1 < R_2$ such that

$$\lambda_1(B(R_1)) < \lambda_1(B(R_2)\setminus\overline{B(R_1)}) < \lambda_2(B(R_1)).$$

Furthermore we let $N \in \mathbf{N}$, and let $\{x_1, ..., x_N\} \subset \{x \in \mathbf{R}^D \mid |x| = R_1\}$. Then we let $\varepsilon \in \mathbf{R}_+ \cup \{0\}$ and define

$$\Omega_\varepsilon = (B(R_2)\setminus\overline{B(R_1)}) \cup \left(\bigcup_{j=1}^N B(x_j, \varepsilon)\right) \cup B(R_1).$$

As a measure of the distance between the points we introduce:

$$\delta = \delta(x_1, ..., x_N) = \inf\left\{\delta > 0 \mid \{x \in \mathbf{R}^D \mid |x| = R_1\} \subset \bigcup_{j=1}^N B(x_j, \delta)\right\}.$$

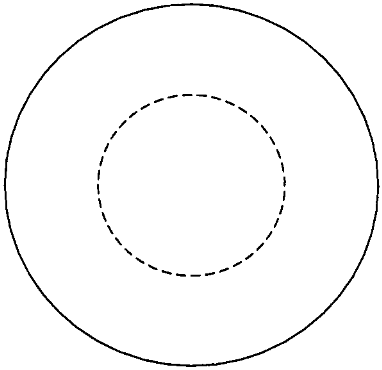

FIG. 1. A typical domain.

This measures the maximal distance between neighboring points. Let us also introduce the simpler

$$\rho = \rho(x_1, ..., x_N) = \min_{j \neq k} \{\mathrm{dist}_{\mathbf{S}^{N-1}}(x_j, x_k)\},$$

where $\mathrm{dist}_{\mathbf{S}^{N-1}}$ is geodesic distance on the sphere. We will always assume that $\rho > 0$ and that $\varepsilon < \rho/2$. To make sure that the holes are evenly distributed, we assume $\delta/\rho \leqslant c_0$, where c_0 is some constant which will be assumed fixed throughout the paper. See Fig. 1 for a typical domain.

We will simplify notation by writing $\lambda_{j,\varepsilon}$ and $u_{j,\varepsilon}$ instead of the heavier $\lambda_j(\Omega_\varepsilon)$ and $u_j(\Omega_\varepsilon)$

Notice that $\Omega_0 = (B(R_2) \backslash \overline{B(R_1)}) \cup B(R_1)$, is the union of two disjoint domains and that the Dirichlet Laplacian on this set is explicitly solvable in terms of Bessel functions.

In [10] it was proved that $-\Delta_\varepsilon$ converges to $-\Delta_0$ in strong resolvent sense as $\varepsilon \searrow 0$. Thus in particular

$$\lambda_{j,\varepsilon} \nearrow \lambda_{j,0}.$$

From this we see that when ε is sufficiently small, then $\lambda_{2,\varepsilon}$ is a simple eigenvalue. We have the freedom to multiply any eigenfunction by a scalar. We will now fix the choice of these scalars for the lowest eigenfunctions of Ω_0: We choose $u_{1,0}$ to be the non-negative Dirichlet groundstate of $B(R_1)$ extended by 0 on $B(R_2) \backslash \overline{B(R_1)}$ and in the same way $u_{2,0}$ as the non-negative Dirichlet groundstate of $B(R_2) \backslash \overline{B(R_1)}$ extended by zero on $B(R_1)$. Notice, that this natural choice of $u_{1,0}$ and $u_{2,0}$ makes them non-negative. That will be important in the arguments below.

Define $\mathcal{N}(u_{2,\varepsilon}) = \overline{\{x \in \Omega_\varepsilon \mid u_{2,\varepsilon}(x) = 0\}}$, then the result of this paper is:

THEOREM 1.1. *Let* $c_0 > 1$, *then there exists* $\delta_0 > 0$ *such that for all choices of* N, $x_1, ..., x_N$ *with* $|x_j = R_1$ *satisfying*

- $\delta(x_1, ..., x_N) \leqslant \delta_0$,
- $\delta(x_1, ..., x_N)/\rho(x_1, ..., x_N) \leqslant c_0$,

then

$$\mathcal{N}(u_{2,\varepsilon}) \cap \partial\Omega_\varepsilon = \emptyset,$$

for all ε *sufficiently small.*

Remark 1.1. Thus, the theorem says, that if we cut many, small holes, and they are almost evenly distributed over the sphere, then the nodal surface will be closed.

All constants are given explicitely and one can therefore give an upper bound on the minimal number of holes necessary for the theorem to hold. This upper bound will be of the order 10^9, which is probably by far too large. Therefore no details the calculation leading to this number will be given.

We use [1] and [9] as standard references for results on stochastic processes. In those books references to the original articles can be found.

2. PRELIMINARY ESTIMATES

The most important result in this section is the following estimate:

LEMMA 2.1. $\exists C > 0$ *independent of* ε, δ *such that*

$$|\langle u_{j,\varepsilon}, u_{2,0}\rangle| \leqslant C \frac{\lambda_{j,\varepsilon}^{1+D/4}}{|\lambda_{j,\varepsilon} - \lambda_{2,0}|} N\varepsilon^D + O(\varepsilon^{D+1}),$$

when ε *is sufficiently small (dep. on* δ*). Here* C *can be chosen as*

$$C = (\partial_r u_{2,0}|_{r=R_1})\, \sigma_{D-1} e^{1+1/8\pi} C_0,$$

where C_0 *is the constant given in Lemma* 2.5 *below, and* $\sigma_{D-1} = \mathrm{vol}_{\mathbf{R}^{D-1}}(B(1))$.

Remark 2.1. $\delta^{-(D-1)}$ is proportional to the number of holes, so N in the above Lemma can be changed to $\delta^{-(D-1)}$, up to a change in the constant C.

The proof of Lemma 2.1 is given in the rest of this section as a series of lemmas.

LEMMA 2.2.

$$\langle u_{j,\varepsilon}, u_{2,0}\rangle = -\frac{\partial_r u_{2,0}|_{r=R_1}}{\lambda_{j,\varepsilon}-\lambda_{2,0}} \int_{S_\varepsilon} u_{j,\varepsilon}(y)\, d\sigma(y),$$

where $S_\varepsilon = \{x \in \Omega_\varepsilon \mid |x| = R_1\}$ *and* σ *is surface measure on the sphere* $\{|x| = R_1\}$.

Proof. This is, in fact, just Green's identity:

$$\begin{aligned}\lambda_{j,\varepsilon}\langle u_{j,\varepsilon}, u_{2,0}\rangle &= \langle -\Delta_\varepsilon u_{j,\varepsilon}, u_{2,0}\rangle \\ &= \lambda_{2,0}\,\langle u_{j,\varepsilon}, u_{2,0}\rangle - \int_{S_\varepsilon} u_{j,\varepsilon}(y)\, \partial_r u_{2,0}(y)\, d\sigma(y).\end{aligned}$$

Now we use that $u_{2,0}$ is rotationally symmetric to reach the conclusion. ∎

Thus we need to estimate $\int_{S_\varepsilon} u_{j,\varepsilon}(y)\, d\sigma(y)$.
Notice the following argument:

LEMMA 2.3. $\forall y \in \Omega_\varepsilon$ *we have*

$$|u_{j,\varepsilon}(y)| \leqslant \|u_{j,\varepsilon}\|_\infty\, e\lambda_{j,\varepsilon}\, \mathbf{E}^y[\tau_\varepsilon],$$

where τ_ε *is the exit time of Brownian motion from* Ω_ε; *i.e.*

$$\tau_\varepsilon = \inf\{t > 0 \mid W_t \notin \Omega_\varepsilon\},$$

where W_t *is D-dimensional Brownian motion and* $\mathbf{E}$ *denotes the expectation.*

Proof.

$$\begin{aligned}u_{j,\varepsilon}(y) &= e^{t\lambda_{j,\varepsilon}}(e^{t\Delta_\varepsilon} u_{j,\varepsilon})(y) \\ &= e^{t\lambda_{j,\varepsilon}}\mathbf{E}^y[u_{j,\varepsilon}(W_t),\, t < \tau_\varepsilon] \\ &\leqslant e^{t\lambda_{j,\varepsilon}}\, \|u_{j,\varepsilon}\|_\infty\, \mathbf{E}^y[1 < \tau_\varepsilon/t] \\ &\leqslant e^{t\lambda_{j,\varepsilon}}\, \|u_{j,\varepsilon}\|_\infty\, \mathbf{E}^y[\tau_\varepsilon]/t.\end{aligned}$$

Now we put $t = \lambda_{j,\varepsilon}^{-1}$ to get the lemma.

LEMMA 2.4.

$$\|u_{j,\varepsilon}\|_\infty \leqslant e^{1/(8\pi)}(\lambda_{j,\varepsilon})^{D/4}.$$

Proof. This was proved in [2, p. 63]. I am grateful to T. Hoffmann-Ostenhof for pointing my attention to this reference. ∎

Thus we will prove a bound $\mathbf{E}^y[\tau_\varepsilon] = O(\varepsilon)$ when $y \in S_\varepsilon$. This will finish the proof of Lemma 2.1.

LEMMA 2.5. *Let $W_t = (X_t^1, X_t^2, ..., X_t^D)$ be D-dimensional Brownian motion. Let*

$$\tau_\varepsilon = \inf\{t > 0 \mid W_t \notin \Omega_\varepsilon\}.$$

Then $\exists C_0 > 0$ such that $\forall \delta > 0$

$$\sup_{y \in S_\varepsilon} \mathbf{E}^y[\tau_\varepsilon] \leqslant C_0 \varepsilon + O(\varepsilon),$$

when ε is sufficiently small (depending on δ). Here C_0 can be chosen as

$$C_0 = \frac{(D-2)\,D}{\sqrt{D-1}} \frac{R_1^{-(D-1)}}{R_1^{-(D-2)} - R_2^{-(D-2)}} (R_2^2 - R_1^2)(1 + R_2/R_1),$$

for $D \geqslant 3$ and

$$C_0 = \frac{1}{2} (R_2^2 - R_1^2)\, R_1^{-1} \frac{1 + R_2/R_1}{\log(R_2/R_1)},$$

for $D = 2$.

Proof. Let $M_t = W_t^2 - Dt$, then M_t is a martingale since the different coordinates of W_t are independent 1-dimensional Brownian motions. Let $x \in S_\varepsilon$, we may choose the coordinates so that $x = (R_1, 0, ..., 0)$. Now, by the martingale property,

$$\mathbf{E}^x[M_t] = \mathbf{E}^x[M_0] = R_1^2,$$

so

$$\begin{aligned} D\mathbf{E}^x[\tau_\varepsilon] &= \mathbf{E}^x[W_{\tau_\varepsilon}^2] - R_1^2 \\ &= R_1^2 \mathbf{P}^x[|W_{\tau_\varepsilon}| = R_1] + R_2^2 \mathbf{P}^x[|W_{\tau_\varepsilon}| = R_2] - R_1^2 \\ &= (R_2^2 - R_1^2)\, \mathbf{P}^x[|W_{\tau_\varepsilon}| = R_2]. \end{aligned}$$

We will now prove that when ε is sufficiently small, then

$$\mathbf{P}^x[|W_{\tau_\varepsilon}| = R_2] \leqslant C_1 \varepsilon + (\tfrac{1}{2} + O(\varepsilon)) \sup_{y \in S_\varepsilon} \mathbf{P}^y[|W_{\tau_\varepsilon}| = R_2].$$

Since $x \in S_\varepsilon$ was arbitrary, this proves the lemma, with $C_0 = 2((R_2^2 - R_1^2)/D)\, C_1$.

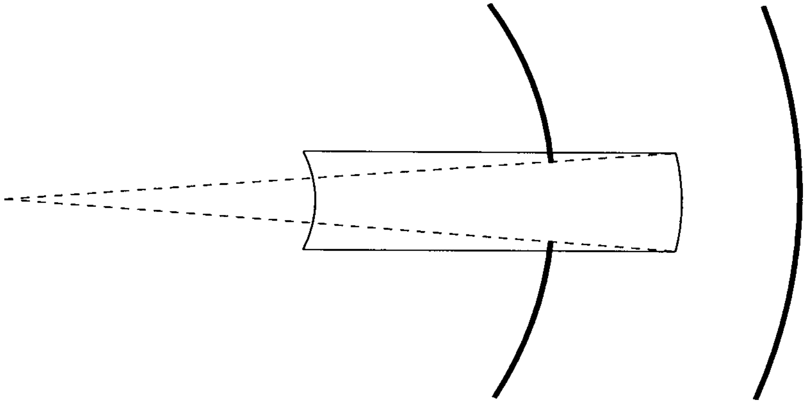

FIG. 2. The stopping time τ_1^ε.

We introduce the following stopping times:
Let

$$k = \frac{\varepsilon}{\sqrt{R_1^2 - \varepsilon^2}} \frac{R_1 + R_2}{2} \approx \frac{R_1 + R_2}{2R_1} \varepsilon,$$

and define

$$\tau_1^\varepsilon = \inf \left\{ t > 0 \,\middle|\, \sqrt{\sum_{j=2}^{D} (X_t^j)^2} = k \text{ or } |X_t^1| = \frac{R_1 + R_2}{2} \text{ or } |X_t^1| = R_1/2 \right\}.$$

(see also Fig. 2).

Define furthermore

$$\tau_1 = \inf\{ t > 0 \mid |W_t| = R_1 \},$$

$$\tau_2 = \inf\{ t > 0 \mid |W_t| = R_2 \}.$$

Then

$$\begin{aligned} \mathbf{P}^x[\,|W_{\tau_\varepsilon}| = R_2] &= \mathbf{P}^x[\,|W_{\tau_\varepsilon}| = R_2 \ \wedge \ \tau_1^\varepsilon < \tau^\varepsilon] \\ &\leqslant \mathbf{P}^x[\,|W_{\tau_\varepsilon}| \circ \theta_{\tau_1^\varepsilon} = R_2] \\ &= \mathbf{E}^x[\mathbf{P}^{W_{\tau_1^\varepsilon}}[\,|W_{\tau_\varepsilon}| = R_2]], \end{aligned}$$

where we used the strong Markov property of Brownian motion. We continue the calculation:

$$
\begin{aligned}
\mathbf{E}^x[\mathbf{P}^{W_{\tau_1^\varepsilon}}[|W_{\tau_\varepsilon}|=R_2]] &= \mathbf{E}^x[\mathbf{P}^{W_{\tau_1^\varepsilon}}[|W_{\tau_\varepsilon}|=R_2,\, \tau_1<\tau_2,\, W_{\tau_1}\in S_\varepsilon]] \\
&\quad + \mathbf{E}^x[\mathbf{P}^{W_{\tau_1^\varepsilon}}[\tau_1>\tau_2]] \\
&\equiv a+b.
\end{aligned}
$$

Let us look at the first term a. Below we will once again use the strong Markov property of Brownian motion.

$$
\begin{aligned}
&\mathbf{P}^{W_{\tau_1^\varepsilon}}[|W_{\tau_\varepsilon}|=R_2,\, \tau_1<\tau_2,\, W_{\tau_1}\in S_\varepsilon] \\
&\quad = \mathbf{P}^{W_{\tau_1^\varepsilon}}[|W_{\tau_\varepsilon}|\circ\theta_{\tau_1}=R_2,\, \tau_1<\tau_2,\, W_{\tau_1}\in S_\varepsilon] \\
&\quad = \mathbf{E}^{W_{\tau_1^\varepsilon}}[1_{\{\tau_1<\tau_2\}}\, 1_{\{W_{\tau_1}\in S_\varepsilon\}}\, \mathbf{P}^{W_{\tau_1}}[|W_{\tau_\varepsilon}|=R_2]] \\
&\quad \leqslant (\sup_{y\in S_\varepsilon} \mathbf{P}^y[|W_{\tau_\varepsilon}|=R_2])\, \mathbf{E}^{W_{\tau_1^\varepsilon}}[1_{\{\tau_1<\tau_2\}}\, 1_{\{W_{\tau_1}\in S_\varepsilon\}}] \\
&\quad \leqslant (\tfrac{1}{2}+O(\varepsilon))(\sup_{y\in S_\varepsilon} \mathbf{P}^y[|W_{\tau_\varepsilon}|=R_2]),
\end{aligned}
$$

when ε is sufficiently small (dep. on δ). Here we used that due to symmetry the chance of "falling back" into the hole we came from, is $\leqslant 1/2$, and the probability of falling into another hole is $O(\varepsilon)$ as $\varepsilon\to 0$ (this follows from [9, Theorem 3.1, p. 102]).

Thus we only need an estimate of order ε of the term b. This is easily accomplished:

First for $D\geqslant 3$. Since $|W_t|^{-(D-2)}$ is a martingale and leaves $[R_2^{-(D-2)}, R_1^{-(D-2)}]$ with probability one, we get ([1, Corollary 4.10, p. 33])

$$
\mathbf{P}^{W_{\tau_1^\varepsilon}}[\tau_1>\tau_2] = \frac{R_1^{-(D-2)} - |W_{\tau_1^\varepsilon}|^{-(D-2)}}{R_1^{-(D-2)} - R_2^{-(D-2)}}\, 1_{\{|W_{\tau_1^\varepsilon}|>R_1\}}.
$$

Therefore, by a first order Taylor expansion,

$$
\begin{aligned}
b &\leqslant c\,\mathbf{E}^x[|X^1_{\tau_1^\varepsilon} - R_1|] \\
&\leqslant c\sqrt{\mathbf{E}^x[|X^1_{\tau_1^\varepsilon} - R_1|^2]} \\
&= c\,\mathbf{E}^x[\sqrt{\tau_1^\varepsilon}],
\end{aligned}
$$

by the Jensen inequality, since $(X^1_t)^2 - t$ is a martingale. Here c can be chosen (up to errors of higher order in ε) as

$$
c = \frac{(D-2)\, R_1^{-(D-1)}}{R_1^{-(D-2)} - R_2^{-(D-2)}}.
$$

Now $\tau_1^\varepsilon \leqslant \tilde{\tau}_1^\varepsilon$, where $\tilde{\tau}_1^\varepsilon = \inf\{t > 0 \mid \sqrt{\sum_{j=2}^D (X_t^j)^2} = k\}$. Remember that $x = (R_1, 0, ..., 0)$. By scaling

$$\mathbf{E}^x[\tilde{\tau}_1^\varepsilon] = k^2 \mathbf{E}^x[\tau],$$

where $\tau = \inf\{t > 0 \mid \sqrt{\sum_{j=2}^D (X_t^j)^2} = 1\}$.

For $D = 2$ we have to use $\log |W_t|$ instead of $|W_t|^{-(D-2)}$.

$$\begin{aligned}\mathbf{P}^{W_{\tau_1^\varepsilon}}[\tau_1 > \tau_2] &= \frac{\log |W_{\tau_1^\varepsilon}| - \log R_1}{\log(R_2/R_1)} 1_{\{|W_{\tau_1^\varepsilon}| > R_1\}} \\ &\leqslant R_1^{-1} \frac{||W_{\tau_1^\varepsilon}| - R_1|}{\log(R_2/R_1)} 1_{\{|W_{\tau_1^\varepsilon}| > R_1\}} \\ &\leqslant R_1^{-1} \frac{||X^1_{\tau_1^\varepsilon}| - R_1|}{\log(R_2/R_1)} 1_{\{|W_{\tau_1^\varepsilon}| > R_1\}} + O(\varepsilon^2).\end{aligned}$$

Now, $\sum_{j=2}^D (X_t^j)^2 - (D-1)t$ is a martingale, thus

$$\mathbf{E}^x[\tau] = \frac{1}{D-1} \mathbf{E}^x \left[\sum_{j=2}^D (X_\tau^j)^2 \right] = \frac{1}{D-1},$$

and we get

$$C_1 = \frac{D-2}{\sqrt{D-1}} \frac{R_1^{-(D-1)}}{R_1^{-(D-2)} - R_2^{-(D-2)}} \frac{1 + R_2/R_1}{2},$$

for $D \geqslant 3$, and

$$C_1 = \frac{R_1^{-1}}{\log(R_2/R_1)} \frac{1 + R_2/R_1}{2},$$

for $D = 2$.

3. PROOF OF THE MAIN THEOREM

Here we will prove the following, more precise, version of the main theorem. Let us first fix the relative magnitude of the various parameters: Since $\rho \approx \delta$, $N \approx \delta^{-(D-1)}$ and we can express everything in terms of δ.

THEOREM 3.1. *$\exists \delta_0 > 0$ $\forall \delta < \delta_0$ $\exists \varepsilon > 0$ such that $u_{2,\varepsilon}(x) > 0$ $\forall x$ with $|x| = R_1 - \delta$.*

Remark 3.1. The following argument shows that Theorem 3.1 implies Theorem 1.1:

Since $\lambda_{2,\varepsilon} \nearrow \lambda_{2,0}$ as $\varepsilon \to 0$, we know that from a certain point $\lambda_{2,\varepsilon} > \lambda_{1,0}$. By the scaling relation $\lambda_j(B(R)) = R^{-2}\lambda_j(B(1))$, we find that for small ε we may choose $R_0^\varepsilon < R_1$ such that

$$\lambda_1(B(R_0^\varepsilon)) = \lambda_{2,\varepsilon}.$$

In the limit we get $R_0^\varepsilon \searrow R_0$, where $\lambda_1(B(R_0)) = \lambda_{2,0}$. Suppose now that $R_0 < R_1 - \delta$ and $\lambda_{2,\varepsilon} > 0\ \forall\ |x| \leqslant R_1 - \delta$. Then, for small ε, either $u_{2,\varepsilon} > 0$ on an open set containing $B(R_0^\varepsilon)$, or $u_{2,\varepsilon}$ takes negative values inside the sphere $\{|x| = R_1 - \delta\}$. In the first case we get $\lambda_{2,\varepsilon} > \lambda_{2,\varepsilon}$ which is obviously a contradiction. Therefore $u_{2,\varepsilon}$ takes negative values inside the sphere $\{|x| = R_1 - \delta\}$ and, by Courant's Nodal Domains Theorem, the nodal surface gets trapped.

The strategy of the proof is the following:
We look at

$$(-\Delta_\varepsilon)^{-(n+1)}\, u_{2,0}(x),$$

where $|x| = R_1 - \delta$. On one hand, we can express this, using the spectral theorem, as a sum of terms of the form

$$\frac{1}{\lambda_{j,\varepsilon}^{n+1}} \langle u_{j,\varepsilon}, u_{2,0}\rangle\, u_{j,\varepsilon}(x),$$

where we expect the term with $j=2$ to be the most important one. If $u_{2,\varepsilon}(x) \leqslant 0$ we get

$$(-\Delta_\varepsilon)^{-(n+1)}\, u_{2,0}(x) \leqslant \left| \sum_{j \neq 2} \frac{1}{\lambda_{j,\varepsilon}^{n+1}} \langle u_{j,\varepsilon}, u_{2,0}\rangle\, u_{j,\varepsilon}(x) \right|,$$

which will be small in a suitable sense, i.e. $O(\varepsilon^D)$.

On the other hand, we can estimate

$$(-\Delta_\varepsilon)^{-(n+1)}\, u_{2,0}(x),$$

from below, rather explicitely, using Brownian motion techniques. This gives a lower bound, which is also of order ε^D. By keeping track of how the constants in both bounds depend on δ, we get a contradiction, for small δ, if $u_{2,\varepsilon}(x) \leqslant 0$. Notice, that it is essential for the lower bound that $u_{2,0}$ is a positive function.

Now we will give the details:

LEMMA 3.1. *Let $y \in S_\varepsilon \equiv \{x \in \Omega_\varepsilon \mid |x| = R_1\}$, and let $d = \operatorname{dist}(y, \partial\Omega_\varepsilon) > 0$. Then $\exists c, C > 0$ independent of ε, δ such that*

$$\frac{1}{\lambda_{2,\varepsilon}^{n+1}} \langle u_{2,\varepsilon}, u_{2,0} \rangle \, u_{2,\varepsilon}(y) \geqslant cd - C\varepsilon^2 - C\varepsilon^{D+1}\delta^{-1}.$$

Here c can be chosen as $c = 1/\lambda_{2,0}^{n+1} \frac{1}{8} \partial_r u_{2,0}|_{r=R_1}$

Proof. Let F be a box around y with sidelength d with two sides at right angles to the vector from y to the origin. Let F_1 be the side of F with the largest distance to the origin. We define the stopping time τ_F as the exit time from F i.e.

$$\tau_F(\omega) = \inf\{t \geqslant 0 \mid W_t(\omega) \notin F\}.$$

Let also

$$\tau_0(\omega) = \inf\{t \geqslant 0 \mid W_t(\omega) \notin \Omega_0\}$$

$$\tau_\varepsilon(\omega) = \inf\{t \geqslant 0 \mid W_t(\omega) \notin \Omega_\varepsilon\}.$$

Notice that $\tau_F \leqslant \tau_0 \leqslant \tau_\varepsilon$. We look at the iterated resolvent $(-\Delta_\varepsilon)^{-(n+1)}$ for a sufficiently big n:

$$[(-\Delta_\varepsilon)^{-(n+1)} u_{2,0}](y) = \sum_{j=1}^{\infty} \frac{1}{\lambda_{j,\varepsilon}^{n+1}} \langle u_{2,0}, u_{j,\varepsilon} \rangle \, u_{j,\varepsilon}(y).$$

From Section 2 we get:

$$\left| \sum_{j \neq 2} \frac{1}{\lambda_{j,\varepsilon}^{n+1}} \langle u_{j,\varepsilon}, u_{2,0} \rangle \, u_{j,\varepsilon}(y) \right| \leqslant C\varepsilon^{D+1} N.$$

We apply the following elementary formula to the resolvent

$$\int_0^\infty t^n e^{-t\lambda} \, dt = \frac{1}{c_n} \frac{1}{\lambda^{1+n}},$$

where c_n is a normalisation. Below, we will repeatedly use the fact that $u_{2,0} \geqslant 0$. We will write τ instead of τ_ε.

$$\begin{aligned}
[(-\Delta_\varepsilon)^{-(n+1)} u_{2,0}](y) &= c_n \int_0^\infty t^n (e^{t\Delta_\varepsilon} u_{2,0})(y)\, dt \\
&= c_n \mathbf{E}^y \left[\int_0^\infty t^n u_{2,0}(W_{t\wedge\tau})\, dt \right] \\
&\geqslant c_n \mathbf{E}^y \left[\int_{\tau_F}^\infty t^n u_{2,0}(W_{t\wedge\tau})\, dt \right] \\
&= c_n \mathbf{E}^y \left[\int_0^\infty (t+\tau_F)^n\, u_{2,0}(W_{(t+\tau_F)\wedge\tau})\, dt \right] \\
&\geqslant c_n \int_0^\infty t^n \mathbf{E}^y [u_{2,0}(W_{(t+\tau_F)\wedge\tau})\Big]\, dt \\
&= c_n \int_0^\infty t^n \mathbf{E}^y [\mathbf{E}^y [u_{2,0}(W_{(t+\tau_F)\wedge\tau}) \,|\, \mathscr{F}_{\tau_F}]]\, dt \\
&= c_n \int_0^\infty t^n \mathbf{E}^y [\mathbf{E}^{W_{\tau_F}} [u_{2,0}(W_{t\wedge\tau})]]\, dt \\
&\geqslant c_n \mathbf{E}^y \left[\mathbf{E}^{W_{\tau_F}} \left[\int_0^{\tau_0} t^n u_{2,0}(W_{t\wedge\tau})\, dt \right]\right] \\
&= \mathbf{E}^y [(-\Delta_0)^{-(n+1)} u_{2,0}(W_{\tau_F})] \\
&= \frac{1}{\lambda_{2,0}^{n+1}} \mathbf{E}^y [u_{2,0}(W_{\tau_F})] \\
&\geqslant \frac{1}{\lambda_{2,0}^{n+1}} \frac{1}{4} \min\{u_{2,0}(z) \,|\, z \in F_1\} \\
&\geqslant \frac{1}{\lambda_{2,0}^{n+1}} \frac{1}{4} \frac{d}{2} \partial_r u_{2,0} \Big|_{r=R_1} + O(\varepsilon^2). \qquad \blacksquare
\end{aligned}$$

Remark 3.2. Let us look at one of the holes

$$H = \{|y| = R_1\} \cap B(x_k, \varepsilon).$$

Let $d(y) = \operatorname{dist}(y, \partial\Omega_\varepsilon)$, and let $d\sigma$ be *normalised* surface measure on $\{|y| = R_1\}$, then

$$\begin{aligned}
\int_H d(y)\, d\sigma(y) &\geqslant \int_{\{y\in H \,|\, d(y)\geqslant\varepsilon/2\}} \varepsilon/2\, d\sigma(y) \\
&= \varepsilon/2(\varepsilon/2)^{D-1} \frac{\sigma_{D-1}}{R_1^{D-1}\tilde\sigma_D} + O(\varepsilon^{D+1}) \\
&= 2^{-D}\varepsilon^D \frac{\sigma_{D-1}}{R_1^{D-1}\tilde\sigma_D} + O(\varepsilon^{D+1}),
\end{aligned}$$

where $\sigma_m = \operatorname{vol}_{\mathbf{R}^m}(B(1))$ and $\tilde\sigma_D$ is the surface measure of $\{|y| = 1\}$ in $\mathbf{R}^D$.

LEMMA 3.2. *Let $x \in \Omega_\varepsilon$, $|x| = R_1 - \delta$. Then there exists a positive constant c_1 such that*

$$[(-\Delta_\varepsilon)^{-(n+1)} u_{2,0}](x) \geqslant c_1 \varepsilon^D \delta^{-(D-1)} + O(\varepsilon^{D+1}).$$

Here c_1 can be chosen as

$$c_1 = 2^{-(2D+3)} \frac{\sigma_{D-1}}{\tilde{\sigma}_D} R_1^{-1}(2R_1 - \delta) \frac{1}{\lambda_{2,0}^{n+1}} (\partial_r u_{2,0}|_{r=R_1}).$$

Proof. The same calculus as above, again using the strong Markov property of Brownian motion, proves that

$$[(-\Delta_\varepsilon)^{-(n+1)} u_{2,0}](x) \geqslant c_n \int_0^\infty t^n \mathbf{E}^x[\mathbf{E}^{W_{\tau_B}}[u_{2,0}(W_{t \wedge \tau})]]\, dt,$$

where

$$\tau_B(\omega) = \inf\{t \geqslant 0 \mid |W_t(\omega)| \geqslant R_1\}$$

is the exit time from the ball of radius R_1. Using the integral formula and the spectral theorem, the right hand side is equal to

$$\mathbf{E}^x \left[\sum_{j=1}^{\infty} \frac{1}{\lambda_{j,\varepsilon}^{n+1}} \langle u_{2,0}, u_{j,\varepsilon} \rangle\, u_{j,\varepsilon}(W_{\tau_B}) \right].$$

Now, the exit distribution of the ball for Brownian motion started at x is explicitly known [1, p. 92] so we get

$$\begin{aligned}
&[(-\Delta_\varepsilon)^{-(n+1)} u_{2,0}](x) \\
&\quad \geqslant \sum_{j=1}^{\infty} \frac{1}{\lambda_{j,\varepsilon}^{n+1}} \langle u_{2,0}, u_{j,\varepsilon} \rangle \int_{S_\varepsilon} K(x, y)\, u_{j,\varepsilon}(y)\, d\sigma(y) \\
&\quad = \frac{1}{\lambda_{2,\varepsilon}^{n+1}} \langle u_{2,0}, u_{2,\varepsilon} \rangle \int_{S_\varepsilon} K(x, y)\, u_{2,\varepsilon}(y)\, d\sigma(y) + O(\varepsilon^{D+1}).
\end{aligned}$$

where $d\sigma$ is *normalised* surface measure on S_ε and

$$K(x, y) = R_1^{D-2} \frac{R_1^2 - |x|^2}{|y - x|^D}.$$

If we just include the hole nearest to x in the integral, we get:

$$\int_{S_\varepsilon} K(x, y) \frac{1}{\lambda_{2,\varepsilon}^{n+1}} \langle u_{2,0}, u_{2,\varepsilon} \rangle u_{2,\varepsilon}(y)\, d\sigma(y)$$

$$\geqslant \left(R_1^{D-2} \frac{2R_1 - \delta}{(\delta+\delta)^D} \delta \right) \left(2^{-D} \varepsilon^D \frac{\sigma_{D-1}}{R_1^{D-1} \tilde{\sigma}_D} \right) \left(\frac{1}{\lambda_{2,0}^{n+1}} \frac{1}{8} \partial_r u_{2,0} \bigg|_{r=R_1} \right)$$

$$+ O(\varepsilon^{D+1}).$$

Thus c_1 can be chosen as

$$c_1 = 2^{-(2D+3)} \frac{\sigma_{D-1}}{\tilde{\sigma}_D} R_1^{-1} (2R_1 - \delta) \frac{1}{\lambda_{2,0}^{n+1}} (\partial_r u_{2,0}|_{r=R_1}). \quad \blacksquare$$

Now, we can prove Theorem 3.1:

Proof. If $|x| = R_1 - \delta$ and $\delta \gg \varepsilon$, then

$$|u_{j,\varepsilon}(x)| \leqslant \|u_{j,\varepsilon}\|_\infty \, e\lambda_{j,\varepsilon} \mathbf{E}^x[\tau_\varepsilon]$$

$$= \|u_{j,\varepsilon}\|_\infty \, e\lambda_{j,\varepsilon} o_\delta(1),$$

where $o_\delta(1)$ tends to zero as δ gets small[2]. Therefore

$$\left| \sum_{j \neq 2} \frac{1}{\lambda_{j,\varepsilon}^{n+1}} \langle u_{j,\varepsilon}, u_{2,0} \rangle u_{j,\varepsilon}(x) \right| \leqslant \varepsilon^D \delta^{-(D-1)} o_\delta(1).$$

On the other hand,

$$[(-\Delta_\varepsilon)^{-(n+1)} u_{2,0}](x) \geqslant c_1 \varepsilon^D \delta^{-(D-1)} + O(\varepsilon^{D+1}).$$

Choosing now ε, δ sufficiently small, we get a contradiction if $u_{2,\varepsilon}(x) \leqslant 0$. $\blacksquare$

[2] In fact,

$$\begin{aligned} \mathbf{E}^x[\tau_\varepsilon] &= \mathbf{E}^x[\tau_0] + O(\varepsilon) \\ &= \frac{R_1^2 - |x|^2}{D} + O(\varepsilon) \\ &= \frac{2R_1 - \delta}{D} \delta + O(\varepsilon). \end{aligned}$$

ACKNOWLEDGMENTS

The author thanks J. P. Solovej for suggesting this problem and for useful discussions. He also thanks S. E. Graversen for discussions on Brownian motion.

REFERENCES

1. R. F. Bass, "Probabilistic Techniques in Analysis," Springer-Verlag, Berlin/New York, 1995.
2. E. B. Davies, "Heat Kernels and Spectral Theory," Cambridge University Press, Cambridge, UK, 1989.
3. D. Gilbarg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Grundlehren der mathematischen Wissenschaften, 2nd ed., Springer-Verlag, Berlin/New York, 1983.
4. M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili, The nodal line of the second eigenfunction of the Laplacian in $\mathbf{R}^2$ can be closed, *Duke Math. J.* **90** (1998), 631–640.
5. M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili, On the nodal line conjecture, *Contemp. Math.* **217** (1998), 33–48.
6. D. Jerison, The first nodal set of a convex domain, *in* "Essays in Honour of E. Stein" (Fefferman, Fefferman, and Wainger, Eds.), pp. 225–259, Princeton Univ. Press, Princeton, 1995.
7. A. D. Melas, On the nodal line of the second eigenfunction of the Laplacian in $\mathbf{R}^2$, *J. Differential Geometry* **35** (1992), 255–263.
8. L. E. Payne, Isoperimetric inequalities and their applications, *SIAM Rev.* **9** (1967), 453–488.
9. S. C. Port and C. J. Stone, "Brownian Motion and Classical Potential Theory," Academic Press, San Diego, 1978.
10. P. Stollmann, A convergence theorem for Dirichlet forms with applications to boundary value problems with varying domains, *Math. Z.* **219** (1995), 275–287.
11. K. Uhlenbeck, Eigenfunctions of Laplace operators, *Bull. Amer. Math. Soc.* **78** (1972), 1073–1076.
12. S. T. Yau, Open problems in geometry, *in* "Proc. Symp. Pure Math.," Vol. 54, Amer. Math. Soc., Providence, RI, 1993.